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Spin liquids on the Husimi cactus

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Abstract. We have studied the *failure* of 'fluctuation-selection' in a ground state manifold associated with an extensive entropy. The recursive structure of the Husimi cactus permits exact calculation, and thus avoids problems associated with non-perturbative fluctuations. The structure of the barriers separating the ground states can be determined; in all the cases we have examined, we find that the system prefers to remain a spin liquid rather than to 'localize' in an ordered state that breaks the degeneracy of the manifold.

A key feature of frustrated spin systems on a lattice is the enhanced role of short-wavelength fluctuations which can modify the long-distance behaviour [1–5]. Frustration, defined as the inability to minimize the energy of each bond individually, often leads to a highly degenerate ground state manifold. Anisotropic thermal and quantum fluctuations may partially lift this degeneracy, selecting spin states that minimize the local curvature of the associated free energy surface. Villain and coworkers [1] have called this phenomenon 'order from disorder', noting that it often leads to a breaking of the lattice symmetry and the selection of a translationally invariant magnetic state [2, 3]. The resulting long-wavelength behaviour is then very similar to that of an unfrustrated magnet with no qualitatively new features aside from the lifting of the ground state degeneracy. Thus, though in principle the interplay between fluctuations and competing interactions could lead to novel spin phases [6–13], this fluctuation-selection phenomenon tends to favour conventionally ordered states and has therefore been a major obstacle in the pursuit of non-translationally invariant spin ground states.

The nearest-neighbour classical Heisenberg model on a kagomé lattice has a particularly rich ground state degeneracy that is only *partially* lifted by Gaussian fluctuations [14–18]. A local discrete degeneracy remains, and further fluctuation-selection into an ordered phase characterized by the wavevector $\boldsymbol{q} = 2\pi(\frac{2}{3}, \frac{2}{3})$ has been proposed by several groups [15, 19–21]. By contrast, recent Monte Carlo studies of this model indicate that this '$\sqrt{3} \times \sqrt{3}$' state does *not* have the largest Boltzmann weight as $T \to 0$ but instead are consistent with the *absence* of a local moment in the zero-temperature limit [17, 18, 22].

This discrepancy between the analytic and the numerical results is intriguing, and suggests that 'order from disorder' in a ground state manifold associated with an extensive entropy requires further study. Clearly the issue of a global rather than a local free energy minimum is crucial. Possible mixing of the states due to finite barrier

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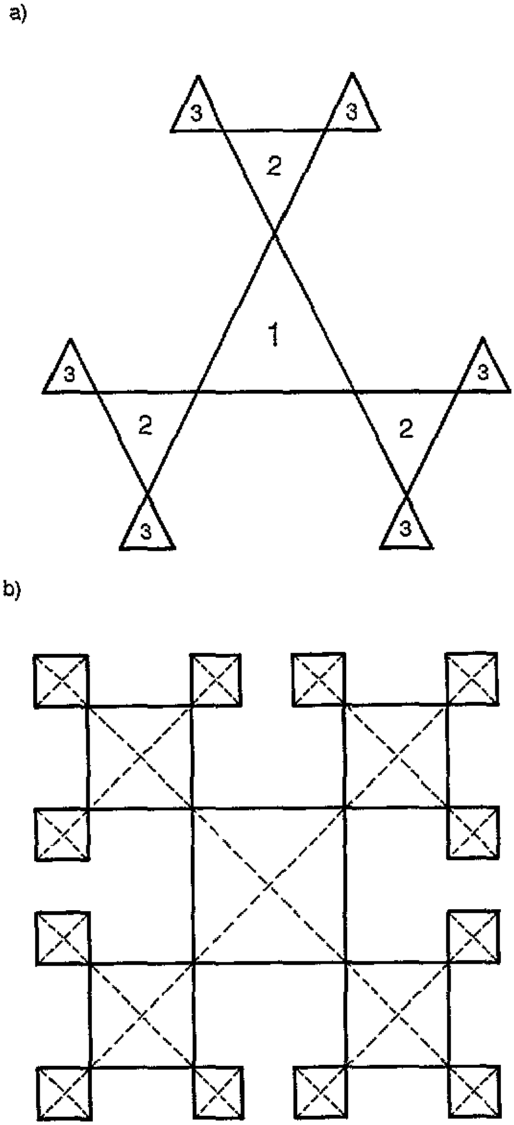

Figure 1. Three generations of (*a*) the triangular and (*b*) the crossed square Husimi cactus.

heights must also be addressed. Finally, a careful accounting of the continuous (S_θ) and the discrete (S_D) entropic terms must be made; fluctuation-selection within the manifold will only occur if

$$S_\theta > S_D \tag{1}$$

which may not be satisfied if S_D is extensive.

These questions are usually difficult to address analytically because of the non-pertubative nature of the fluctuations involved. However, they are tractable on a Husimi cactus (figure 1), a pseudo-lattice with a polygon at each node whose recursive structure permits exact calculations.† We can easily construct a continuous spin model on this pseudo-lattice that has the required extensive ground state entropy, and thus can study the associated 'order from disorder' in a controlled fashion.

We start by considering the classical xy model on the triangular cactus (figure 1(a)) and determine the number of ground states. These states are entropically equivalent and are separated by finite barriers. There is a symmetry-breaking transition at finite temperature to a nematic phase with $\langle e^{3i\theta}\rangle \neq 0$, but there is no conventional ordering and the resulting state is a classical spin liquid. We also discuss the analogous Heisenberg case, where 'order from disorder' again fails to select from within the ground state manifold. Next we turn to a crossed square cactus model (figure 1(b)) with an extensive discrete entropy and *inequivalent* ground states separated by finite barriers. Here we find a clear example where the system prefers to remain 'liquid-like', rather than 'localizing' in an ordered state that breaks the degeneracy of the manifold. Finally, we return to the triangular cactus with the quantum $S=\frac{1}{2}$ problem, and show that it is a 'marginal' dimer liquid with an entropy that scales with the logarithm of the number of sites. All of the systems we have studied here are liquids, and we end with some general comments relating the entropy/site to the barrier heights. The possibility of glassiness in the absence of disorder is also discussed briefly.

We begin with a study of the classical xy antiferromagnet on the triangular cactus (figure 1(a)); it is defined by the Hamiltonian

$$H=\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j) \tag{2}$$

where the summation is over nearest-neighbour bonds. The constraint on each triangular plaquette

$$\sum_{(i\in\Delta)} S_i=0 \tag{3}$$

does not define a *unique* ground state configuration, and each spin is an integer multiple of $2\pi/3$ up to a global continuous rotation. Once this global degree of freedom is fixed, these configurations are isomorphic to the ground states of the three-state Potts model

$$H_{\rm Potts}=\sum_{\langle ij\rangle}\delta(\sigma_i,\sigma_j) \tag{4}$$

on the same lattice where the Potts variables $\sigma_i\in\{0,1,2\}$ are constrained to take different values for nearest-neighbour sites. There exist

$$W=3\times 2^{(N-1)/2} \tag{5}$$

of these Potts ground states where N is the number of lattice sites; thus the ground state manifold has a finite entropy/site $\tilde{S}_0\equiv S_0/k_B=\frac{1}{2}\ln 2\cong 0.347$, comparable to that of the Ising triangular antiferromagnetic [24] ($\tilde{S}_0\cong 0.338$) and approximately twice that associated with the three-state Potts model on a kagomé lattice [25] ($\tilde{S}_0\cong 0.126$).

The full ground state manifold of the three-state Potts model on the triangular

† For a pedagogical review of recursive structures, see [23].

cactus can be explored by performing simple lattice symmetry transformations, in contrast to the case for most Euclidean lattice models that possess an extensive ground state entropy. As displayed in figure 2, a permutation of any two neighbouring sites and the two lattice branches emanating from them transforms one ground state into another. We thus deduce that all the ground states are equivalent, a feature that should persist in the presence of thermal fluctuations.

Does the Potts system freeze into one of these ground states at a finite temperature? A study of finite-temperature spin dynamics is one possible approach to this problem, though it is simpler to remain within the framework of equilibrium statistical mechanics. We note that a ground state configuration on this cactus is completely determined by the spin values at the boundaries, and that some boundary configurations do not correspond to any global ground state. We can thus bias the system towards a particular bulk ground state by applying an infinitesimal symmetry-breaking field at the boundaries and checking for its subsequent amplification. More specifically, if we choose a set of boundary values, denoted by $\bar{\sigma}_i$, compatible with a bulk

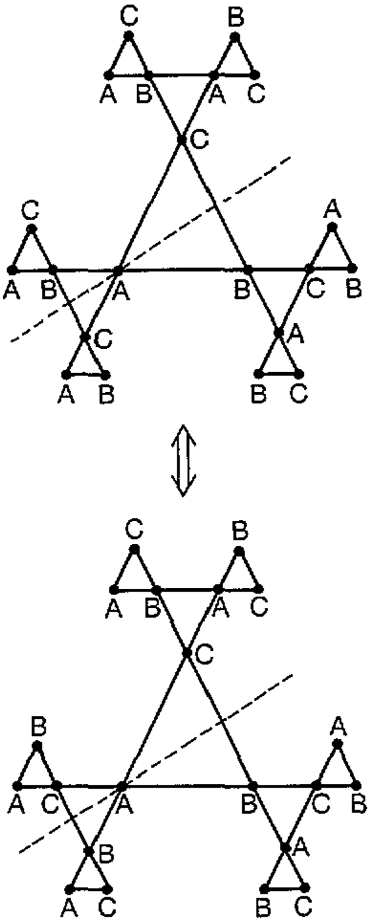

Figure 2. An example of the lattice symmetry operation that transforms one Potts ground state in another, indicating their equivalence.

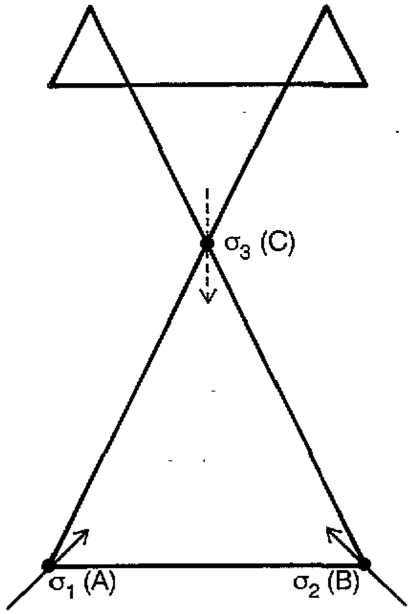

Figure 3. A pictorial representation of an infinitesimal staggered field applied to the boundary spins σ_1 and σ_2; here A and B (full arrows) represent the Potts directions of the applied field and C (broken arrow) is an internal field generated after the trace over the boundary spins has been performed.

ground state a term

$$H_{\text{boundary}} = h \sum_{i}^{(b)} \delta(\sigma_i, \tilde{\sigma}_i) \tag{6}$$

must be added to the original Hamiltonian (4). The partition function is calculated by integrating out these boundary spins. As illustrated schematically in figure 3, we choose the boundary values of the bias field to be $\tilde{\sigma}_1 = \sigma_A$ and $\tilde{\sigma}_2 = \sigma_B$ where $\tilde{\sigma}_3 = \sigma_C$ after $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ have been traced out; here the letters A, B and C serve as alternative labels for the Potts variables. The partial trace over σ_1 and σ_2 yields

$$\tilde{Z}(\sigma_3) = \sum_{\sigma_1, \sigma_2} \exp\{\beta h[\delta(\sigma_1, \sigma_A) + \delta(\sigma_2, \sigma_B)]\} \exp\{-\beta[\delta(\sigma_1, \sigma_2) + \delta(\sigma_2, \sigma_3) + \delta(\sigma_3, \sigma_1)]\} \tag{7}$$

which determines the renormalized symmetry-breaking field by the condition

$$\mathrm{e}^{\beta h'} \equiv \frac{\tilde{Z}(\sigma_3 = \sigma_c)}{\tilde{Z}(\sigma_3 = \sigma_A, \sigma_B)} = \left\{ \frac{\mathrm{e}^{2\beta h} + 4\mathrm{e}^{\beta(h-1)} + 2\mathrm{e}^{-\beta} + 1 + \mathrm{e}^{-3\beta}}{\mathrm{e}^{\beta(2h-1)} + 2\mathrm{e}^{\beta(h-1)} + \mathrm{e}^{\beta h} + 3\mathrm{e}^{-\beta} + \mathrm{e}^{\beta(h-3)} + 1} \right\} \tag{8}$$

so that

$$\left.\frac{\mathrm{d}h'}{\mathrm{d}h}\right|_{h=0} = \frac{1 - \mathrm{e}^{-3\beta}}{2 + 6\mathrm{e}^{-\beta} + \mathrm{e}^{-3\beta}} \equiv f_P(\beta) \tag{9}$$

for small h. We note that, since $f_P(\beta)$ in (9) is an increasing function of β (see figure 4), it is *always* smaller than its asymptotic value as $\beta \to \infty$ ($T \to 0$), so that the renorma-

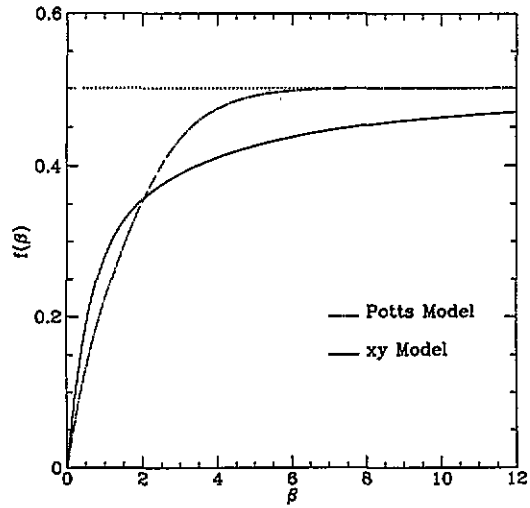

Figure 4. Amplification $f(\beta)$ associated with an $e^{i\theta}$-order parameter for the Potts and the xy models, respectively, on the triangular Husimi cactus.

lized field is always less than half the orginal value ($h' < \frac{1}{2}h$). Physically this implies that h' is not amplified but instead scales to zero in the limit of infinite iteration. Thus we conclude that there is no staggered order associated with the Potts model at any temperature.

Qualitatively we can understand this result by considering the energy barriers separating different ground states. We can identify open paths of alternating spin orientations (ABAB ...) in all the Potts ground state configurations (figure 5) which, in analogy to the kagomé case, will be subsequently denoted as folds [17, 18, 26]. Permutation of the two Potts variables along a fold results in a new ground state; furthermore, the associated energy cost is at most that of two broken bonds. In general it is possible to connect any two Potts ground states by a finite succession of such fold-permutations; therefore, all the barriers between the ground states are finite. As a result, the system remains delocalized in phase space, and can be characterized as a classical spin liquid.

We can extend this analysis to the xy model on the triangular cactus where

$$\tilde{Z}(\theta) = \sum_{m=-\infty}^{m=+\infty} X_m \, e^{im\theta} \tag{10}$$

with the condition that the fields $X_m \ll 1$ for $m \neq 1$ and $X_0 = 1$. In analogy to (8), an integration over two boundary spins θ_1 and θ_2 in the presence of a symmetry-breaking field leads to

$$\tilde{Z}(\theta_3) = \sum_{m_1, m_2} \int d\theta_1 \, d\theta_2 X_{m_1} X_{m_2} \exp[i(m_1\theta_1 + m_2\theta_2)]$$
$$\times \exp[-\beta(\cos(\theta_3 - \theta_1) + \cos(\theta_3 - \theta_2) + \cos(\theta_1 - \theta_2))] \tag{11}$$

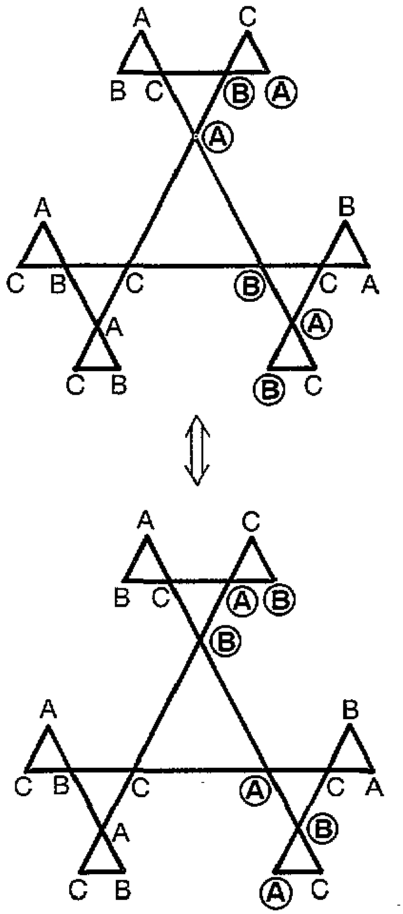

Figure 5. An illustration of two triangular ground states connected by permutations of the Potts variables A and B along a fold.

which becomes

$$\tilde{Z}(\theta_3) = \sum_{m_1, m_2} \int \mathrm{d}\theta_1 \, \mathrm{d}\theta_2 X_{m_1} X_{m_2} \exp[\mathrm{i}(m_1 + m_2)\theta_3] \int \mathrm{d}u \, \mathrm{d}v \exp[\mathrm{i}(m_1 u + m_2 v)]$$
$$\times \exp[-\beta(\cos u + \cos v + \cos(u - v))] \tag{12}$$

where we have made the change of variables $\theta_1 = \theta_3 + u$ and $\theta_2 = \theta_3 + v$. We define

$$G_m \equiv \int \mathrm{d}u \, \mathrm{d}v \exp(\mathrm{i}mu) \exp[-\beta(\cos u + \cos v + \cos(u - v))] \tag{13}$$

and then write

$$\tilde{Z}(\theta_3) = G_0(\beta) \left\{ 1 + 2 \sum_{m \neq 0} X_m \frac{G_m(\beta)}{G_0(\beta)} e^{im\theta_3} + \cdots \right\} \tag{14}$$

which leads to the condition

$$\left.\frac{\mathrm{d}X'_m}{\mathrm{d}X_m}\right|_{X_m=0}=\frac{2G_m(\beta)}{G_0(\beta)}\equiv g_m(\beta) \tag{15}$$

that determines the absence/presence of long-range order associated with a given field. Since $G_m(\beta)=G^*_{-m}(\beta)$ and $G_m(\beta)=G_{-m}(\beta)$ we can rewrite (13) as

$$G_m(\beta)=\int \mathrm{d}u\,\mathrm{d}v\cos(mu)\exp\{-\beta[\cos u+\cos v+\cos(u-v)\}. \tag{16}$$

The Boltzmann weight in (16) is maximized when $(u,v)=\pm(2\pi/3,-2\pi/3)$ so that intuitively $G_m(\beta)$ will be large when m is a multiple of three $((2\pi/3)^m=n\pi)$. We also expect the largest transition temperature $T_c(m)$ for $m=3$ since

$$\lim_{\beta\to\infty}\frac{2G_3(\beta)}{G_0(\beta)}=2 \qquad (>1) \tag{17}$$

and thus will study $g_3(\beta)$. We note that the amplification function $g_1(\beta)$ in (15) corresponds to *ferromagnetic* ordering; the analogous expression $f_1(\beta)$ associated with $e^{i\theta}$ *antiferromagnetic* long-range order (i.e. with three phases modulo $2\pi/3$) is

$$f_1(\beta)=-\frac{G_1(\beta)}{G_0(\beta)} \tag{18}$$

where $G_1(\beta)$ and $G_0(\beta)$ are defined in (16).

The functions $f(\beta)\equiv f_1(\beta)$ and $g(\beta)\equiv g_3(\beta)$ are plotted in figures 4 and 6; however, let us try to develop a qualitative feeling for results. For example, application of an infinitesimal staggered $e^{i\theta}$ symmetry-breaking field ε at the boundaries leads to

$$\tilde{Z}(\theta_3)=\int_{-\pi}^{\pi}\mathrm{d}\theta_1\int_{-\pi}^{\pi}\mathrm{d}\theta_2(1+\varepsilon\cos(\theta_1-\theta_A))(1+\varepsilon\cos(\theta_2-\theta_B))\exp\{-\beta H(\theta_1,\theta_2,\theta_3)\} \tag{19}$$

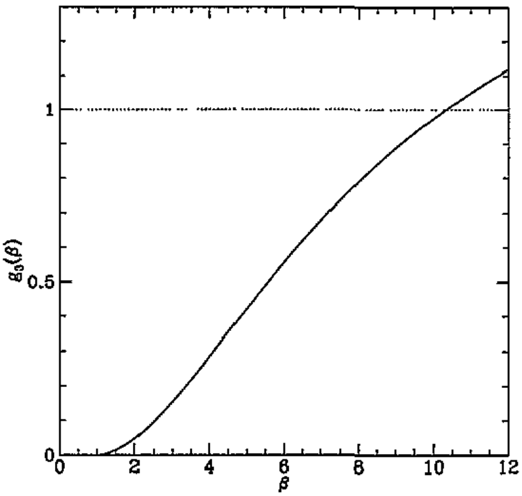

Figure 6. Amplification $g(\beta)$ associated with an $e^{i3\theta}$-order parameter for the xy model on the triangular Husimi cactus.

where we chose $\theta_A=0$, $\theta_B=2\pi/3$ and $\theta_C=4\pi/3$. It is difficult to evaluate $\tilde{Z}(\theta_3)$ in (19) at finite temperatures; however, since our interest is in the presence/absence of a transition, it is sufficient to study the behaviour of a symmetry-breaking field in the zero-temperature limit. This amounts to replacing the Boltzmann weight in (11) by

$$\delta\left(\theta_1-\theta_3+\frac{2\pi}{3}\right)\delta\left(\theta_2-\theta_3-\frac{2\pi}{3}\right)+\delta\left(\theta_1-\theta_3-\frac{2\pi}{3}\right)\delta\left(\theta_2-\theta_3+\frac{2\pi}{3}\right) \quad (20)$$

and in this limit

$$\tilde{Z}(\theta_3, T=0)=2+\varepsilon\cos(\theta_3-\theta_C)+O(e^2) \quad (21)$$

which yields the mapping

$$\varepsilon'\rightarrow\frac{\varepsilon}{2} \quad (22)$$

at each iteration, implying the absence of long-range staggered order at any finite temperature similar to the Potts result (see figure 4).

Broken global rotational symmetry is a clear feature that could distinguish the *xy* from the Potts case, and we next study the propagation of an infinitesimal field associated with $e^{i3\theta}$ order towards the bulk of the system. Analogous to our treatment above, we consider

$$\tilde{Z}'(\theta_3)=\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2(1+\varepsilon\cos 3\theta_1)(1+\varepsilon\cos 3\theta_2)\exp(-\{\beta H(\theta_1,\theta_2,\theta_3)\}. \quad (23)$$

As before, we analyse the behaviour of this symmetry-breaking field in the zero-temperature limit, substituting expression (20) for the Boltzmann weight; we then find that

$$\tilde{Z}'(\theta_3, T=0)=2+4\varepsilon\cos 3\theta+O(\varepsilon^2) \quad (24)$$

so that

$$\varepsilon'\rightarrow 2\varepsilon \quad (25)$$

implying the *presence* of long-range order associated with a finite $\langle e^{i3\theta}\rangle$ and consistent with our previous discussion. Thus at $\beta_c=10.4$ (determined from $g(\beta_c)=1$ in figure 6)) the *xy* spins undergo *partial* ergodicity-breaking associated with the freezing of their continuous phase, but there is *no* selection of a particular Potts state. We note that the bulk partition function can be written as

$$Z=2\pi(Z_\Delta)^{N_t} \quad (26)$$

where Z_Δ is the partition function associated with each of N_t triangles, and thus is always *smoothly* varying with β. The nematic transition is therefore *inaccessible* in the absence of a symmetry-breaking field, which can lead to a *bulk* fixed-point value $h^*(\beta)$ that is a singular function of temperature.

The Heisenberg model on this triangular cactus (figaure 1(*a*)) is also interesting, particularly as its ground states are *inequivalent*. Here the Hamiltonian is

$$H=\sum_{\langle ij\rangle}\boldsymbol{n}_i\cdot\boldsymbol{n}_j \quad (27)$$

where $(\boldsymbol{n}_i)$ is a three-dimensional vector. Most of the ground states are not coplanar;

indeed, starting from a coplanar state it is possible to rotate all the spins beyond a given site i around the vector $\boldsymbol{n}_i$ without an energy cost. The eigenmode spectrum depends on the choice of ground state configuration, a feature that arises from the non-Abelian nature of the underlying symmetry group SO(3). Can 'order from disorder' select a submanifold from this ensemble? In particular, are coplanar states favoured as in the Heisenberg kagomé antiferromagnet? Accordingly we introduce boundary fields that favour an xy plane orientation, and investigate their renormalization after a trace on the boundary spins has been performed. We then have

$$\tilde{Z}(\boldsymbol{n}_3)=\int \mathrm{d}^2\boldsymbol{n}_1\,\mathrm{d}^2\boldsymbol{n}_2(1+\varepsilon\sin^2\theta_1)(1+\varepsilon\sin^2\theta_2)\exp(-\beta H(\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3)) \quad (28)$$

where we have used the usual spherical coordinates (θ,ϕ). In the $T=0$ limit the integral is performed over the triangular ground state configurations keeping a fixed $\boldsymbol{n}_3$, which is parametrized by a rotation angle around $\boldsymbol{n}_3$. A simple calculation yields

$$\tilde{Z}(\boldsymbol{n}_3)=\left\{1+\frac{3}{2}\varepsilon-\frac{\varepsilon}{4}\sin^2\theta_3+O(\varepsilon^2)\right\} \quad (29)$$

up to a normalization constant, so that after iteration we get the mapping

$$\varepsilon'=-\frac{\varepsilon}{4}+O(\varepsilon^4) \quad (30)$$

and the field is not amplified in the bulk of the system. There is thus no phase transition into a coplanar submanifold, and consequently we do not expect a low-temperature phase associated with long-range order in $\langle e^{3i\theta}\rangle$.

An anisotropic Heisenberg interaction, that is, making the replacement

$$\boldsymbol{n}_i\cdot\boldsymbol{n}_j\to n_i^x n_j^x+n_i^y n_j^y+\alpha n_i^z n_j^z \qquad 0\leqslant\alpha\leqslant 1 \quad (31)$$

in (27), would in principle allow us to study the crossover between the Heisenberg and the xy models. For example, for $\alpha<1$, the ground state manifold is characterized by $\theta=\pi/2$ with only coplanar states. Using the same arguments as before, we find that there exists a finite-temperature transition into a state where $\langle e^{3i\theta}\rangle\neq 0$. However, a first-principles calculations of $T_c(\alpha)$ is more delicate since after an iteration the symmetry-breaking field becomes a function of *both* θ and ϕ. As α approaches unity, the fluctuations of θ about $\pi/2$ become large and the subsequent behaviour of the system cannot be described by a single variable iteration.

The Heisenberg model on the triangular cactus has both an extensive entropy and inequivalent ground states, and fluctuation-selection appears to fail. How general is this result? Let us turn to a more tractable system that shares these two features: the xy model on the crossed square Husimi cactus (figure 1(b)). We choose our Hamiltonian to be the sum of elementary plaquette Hamiltonians where

$$H_{\text{plaquette}}=\sum_{1\leqslant i\leqslant j\leqslant 4}(\cos(\theta_i-\theta_j)+\tfrac{1}{2})^2(\alpha-\cos(\theta_i-\theta_j)) \qquad \alpha>1 \quad (32)$$

so that the ground state configurations are isomorphic to those of the three-state Potts model on the same lattice. The plaquette configuration (A,A,B,C) with associated spin variables $(0,0,2\pi/3,4\pi/3)$ is an absolute minimum for $1<\alpha<11/3$.

In order to characterize the ground states of this crossed square model we need to generalize the folds already discussed for the triangular case. We begin by observing

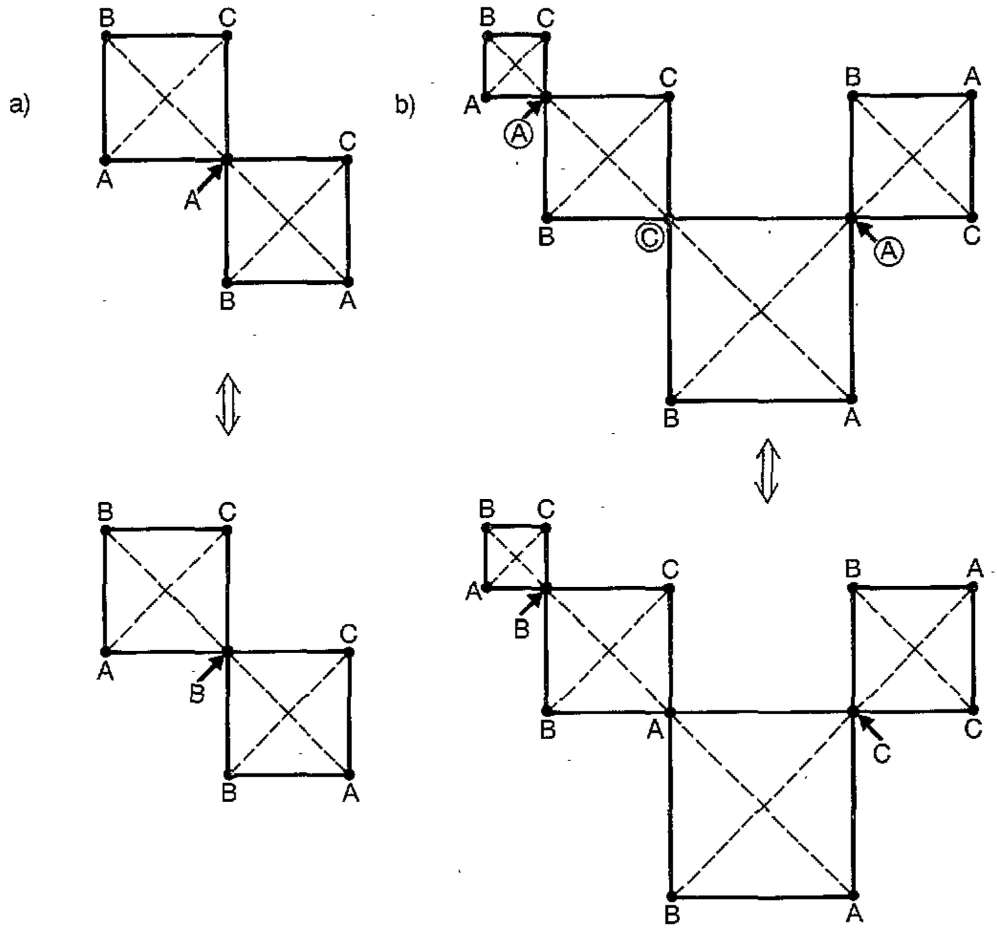

Figure 7. Examples of folds on the crossed square Husimi cactus with length (a) $L=1$ and (b) $L=3$ where the fold ends are indicated by arrows.

that each plaquete has a repeated Potts variable associated with a weak bond. If one of the two corresponding spins is modified, the energy of that particular plaquette *does not change. However, an adjacent square that shares the modified spin may now* be in an excited state, and thus a spin on this second plaquette must flip; this process continues until spin-flips are no longer energetically necessary (see figure 7). If the ground states are equally weighted in the manifold, the fold has equal probability distribution associated with the observation of a fold of length L is

$$P(L)=2^{-(L+1)} \tag{33}$$

where L is an integer. We contrast this situation with that of the triangular case, where all the folds traversed the entire cactus and were only interrupted by the boundaries. For the crossed square cactus the eigenmode spectrum for spin-waves in a given ground-state will depend on the detailed configuration of the weak links, which are the bonds connecting nearest-neighbour sites with the *same* Potts variables in this particular ground state. In particular, a well-defined subset of the ground state manifold can have a fold distribution markedly different from (33). For example, a configuration where the repeated Potts variable is A for all elementary squares only possesses folds with length $L=1$. Heuristically, we can think of these folds as zero modes; thus such a state be the most 'entropically flexible' in the manifold. Is it selected by the fluctuations? We note that this question is of great interest for the closely connected xy kagomé example; there the full ground state manifold has a distribution $P(L)\sim L^{-4/3}$ and the fluctuation-selection of the state with the maximum number of zero modes has not been observed numerically [17, 18, 22].

On the crossed square Husimi cactus, the fluctuation-selection of the $L=1$ state corresponds to the development of staggered order. We proceed with a similar treatment as already described; however, to thoroughly investigate this possibility, we must consider the two cases where the repeated Potts variable is on the bounday and on the interior. For the boundary case we have

$$\tilde{Z}(\theta_4)=\int d\theta_1\, d\theta_2\, d\theta_3\, (1+\varepsilon\cos(\theta_1-\theta_A))$$

$$(1+\varepsilon\cos(\theta_2-\theta_A))\,(1+\varepsilon\cos(\theta_3-\theta_B))\exp(-\beta H(\theta_1,\theta_2,\theta_3,\theta_4)) \qquad (34)$$

which becomes

$$\tilde{Z}(\theta_4)=12+\varepsilon\{6\cos(\theta-\theta_C)+3\cos(\theta-\theta_B)\}+O(\varepsilon^2) \qquad (35)$$

in the limit $\beta\to\infty$, and there is *no* enhancement of the symmetry-breaking term ($\varepsilon'=\varepsilon/2$). Otherwise, the repeated Potts variable could be on an inner site with

$$\tilde{Z}(\theta_4)=\int d\theta_1\, d\theta_2\, d\theta_3\, (1+\varepsilon\cos(\theta_1-\theta_A))$$

$$\times(1+\varepsilon\cos(\theta_2-\theta_B))\,(1+\varepsilon\cos(\theta_3-\theta_C))\exp(-\beta H(\theta_1,\theta_2,\theta_3,\theta_4)) \qquad (36)$$

which yields

$$\tilde{Z}(\theta_4)=12+O(\varepsilon^2) \qquad (37)$$

in the $T\to 0$ limit, again indicating the *absence* of an amplified symmetry field.

How do we interpret these results? Within a spin-wave analysis we find that the partition function

$$Z'_{SW}=\frac{1}{N}\prod_{\text{squares}}\int d\theta'_1\, d\theta'_2\, d\theta'_3 \exp\left(-\frac{\beta}{2}H(\theta'_1,\theta'_2,\theta'_3,0)\right) \qquad (38)$$

is *factorizable*, a result that relies on the special topology of the lattice and not on the specific (Gaussian) approximation used; here θ'_i refers to the spin deviations from a given configuration. In any ground state configuration there is just one weak link per elementary square. The spin-wave free energy, the product of *all* the non-vanishing eigenvalues, is *independent* of the details of the positions of these links. More specifically, the factorizability of the free energy implies the *absence* of interactions between the weak bonds and thus there is no fluctuation-selection to all orders in the spin deviations.

How similar is the quantum case? In order to address this question, we turn briefly to the quantum $S=\frac{1}{2}$ Heisenberg spin model on the triangular cactus, and the analysis of the ground state follows closely that of the Majumdar–Ghosh model [27, 28]. We

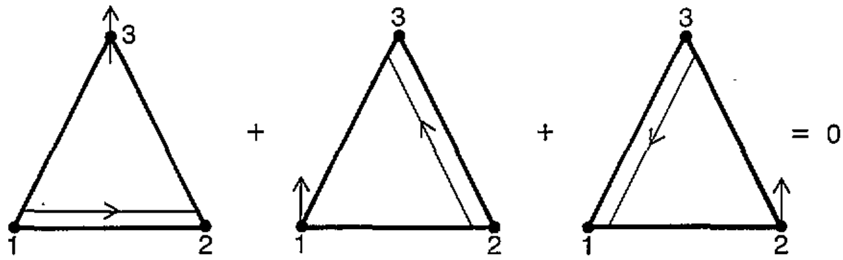

Figure 8. A graphical representation of the linear dependence of the three dimer configurations on an elementary triangle.

first note that the Heisenberg Hamiltonian can be rewritten in the form

$$H=\sum_{\Delta}(S_1+S_2+S_3)^2 \tag{39}$$

up to a constant term where each elementary triangle has a total spin $S=\frac{1}{2}$ in a ground state. We use the usual notation (ij) to denote a singlet bond between sites i and j. The ground state manifold for a single triangle $(1, 2, 3)$ is spanned by the states $(12)\sigma_3$, $(23)\sigma_1$ and $(31)\sigma_2$, where σ_i is the spin value at site i. The six states thus defined are *not* linearly independent since

$$(12)\sigma_3+(23)\sigma_1+(31)\sigma_2=0 \tag{40}$$

if $\sigma_1=\sigma_2=\sigma_3$. Indeed, there only exist two independent $S=\frac{1}{2}$ states on the triangle, and its ground state subspace thus has dimension four. It is straightforward to construct a set of ground states from this representation, as illustrated in figure 8. These states are labelled by $|i, \sigma\rangle$ where i is the site of the free spin with z-component $\sigma=\pm\frac{1}{2}$. A unique dimer configuration is defined by each choice of free spin position. These states are ground states by construction, since each triangle has one singlet bond. By recursion on the number of triangles one can show that the ground states are linear combinations of these $|i, \sigma\rangle$ states; the proof is straightforward but lengthy, so it will not be presented here. We note, though, that all of these states are *not* linearly independent; in fact, for each value of $\sigma=\pm\frac{1}{2}$ we have the constraint

$$|i, \sigma\rangle+|j, \sigma\rangle+|k, \sigma\rangle=0 \tag{41}$$

per triangle where i, j and k are the three triangular sites. Thus we have $2N_s$ dimer states (see figure 9 for an example) with $2N_t$ independent constraints, where N_s and N_t are the number of sites and triangles, respectively. Since $N_t=\frac{1}{2}(N_s-1)$, the dimension of the ground state manifold is

$$W=2(N_s-N_t)=N_s+1. \tag{42}$$

Expression (42) indicates that though the quantum fluctuations are not selecting a unique ground state, they are more efficient than their thermal counterparts [29] in restricting the phase space explored by the system ($\tilde{S}_0=(1/N_s)\ln N_s$). It would certainly be very interesting to characterize the elementary excitations of this model, as well as to reconstruct its low-temperature thermodynamics. Quantum tunnelling between classical states associated with its large S limit [30, 31] could also be studied, and we leave these questions for future work.

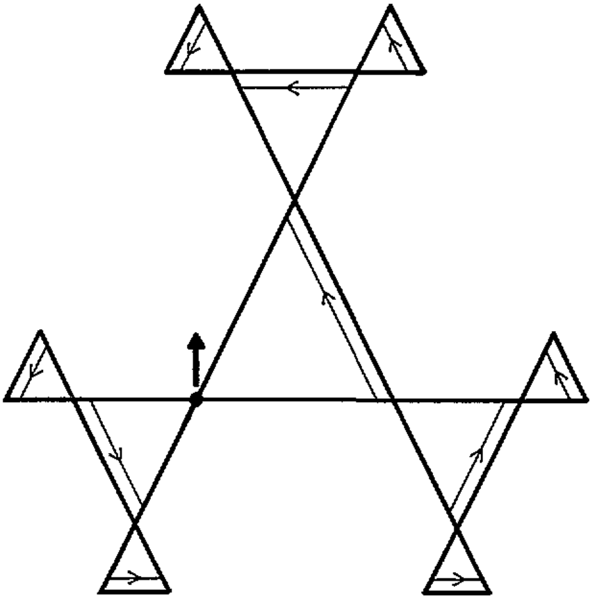

Figure 9. A dimer state on a section of the triangular Husimi cactus with three generations.

In conclusion, we have studied a number of examples where the conventional 'order from disorder' mechanism fails. Each of these spin models has coexisting continuous and discrete low-energy excitations, and a total entropy

$$S = S_D + S_\theta \tag{43}$$

where *both* contributions scale with the site number. Most frustrated spin systems minimize their free energy by 'localizing' in a state of high symmetry associated with the *maximum* continuous entropy. However, the situation in a ground state manifold associated with a large, *extensive* S_D is not straightforward. In analogy to the case of electronic propagation in disordered media, the discrete entropy S_D plays the role of the bandwidth W which prefers 'delocalization'; similarly, the continuous entropy S_θ is like the root mean square of the random part of the potential $\sqrt{\overline{\delta V_r^2}}$ which inhibits diffusion. We thus expect the absence of fluctuation-selection if the condition

$$S_D > S_\theta \tag{44}$$

is satisfied, analogous to the more conventional delocalization criterion

$$W > \sqrt{\overline{\delta V_r^2}}$$

for the electron problem; since S_θ involves the continuous fluctuations of *each* spin about its equilibrium position, clearly S_D must scale with the number of sites for (44) to be fulfilled. We emphasize that the entropy, unlike the electronic energy, has *no* intrinsic time-scale; thus, both a liquid and a glass are 'entropically delocalized' and must be distinguished by their dynamics. In this paper, using the recursive structure of the Husimi cactus, we have studied analytically spin models with very large extensive discrete entropies; we have shown that they do not select 'spin-crystalline' ground states but prefer to remain 'entropically delocalized' in agreement with (44).

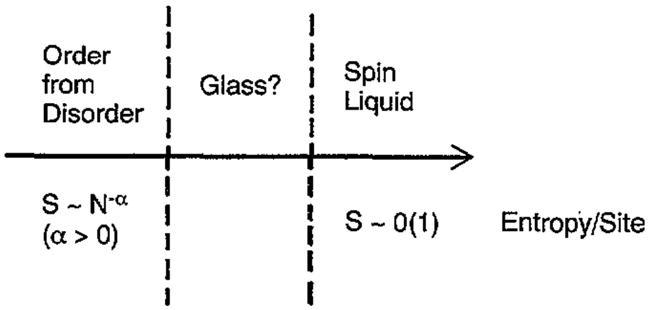

Figure 10. An illustration of the heuristic connection between the entropy/site and the resulting low-temperature phase.

Another common feature of all of the models studied here is the presence of *finite* barriers connecting the ground states at zero temperature. At $T=0$, where these barriers are purely associated with an energy cost, the continuous degrees of freedom provide alternative paths between different ground states and thus these barriers are *lower* than their discrete counterparts. Clearly, entropic considerations must be taken into account at finite temperatures. The main effect of decreasing temperature on the resulting spin liquid is to increase the residence time in each well, but the system does not 'dynamically' localize in phase space as would a glass. In the presence of such a large number of small barriers, the issue of lifting the degeneracy of the ground state loses most of its importance. To this end we have studied two different models with large discrete entropies and finite barriers; though in one case the ground states were equivalent and in the other they were not, the qualitative features of the final liquid states are indistinguishable. We also note that the presence of such a spin liquid is *not* incompatible with a finite-temperature nematic transition; for example, on the kagomé lattice such a transition is accompanied by a jump in the spin stiffness and the binding of Kosterlitz–Thouless vortices.

There has been some hope that the interplay between continuous and discrete degrees of freedom could lead to glassy behaviour in the absence of disorder. This would require the *increase* of the discrete barriers by low-lying continuous excitations, a phenomenon that has not been observed here. Returning to the discrete models, we note the connection between the number of ground states and the typical barriers separating them. Intuitively, an increase in the extensive entropy results in a *decrease* in the typical barrier height, since the associated ground states will be closely packed in phase space. Conventional 'order from disorder' works very well in the limit of vanishing entropy/site, possibly because the barriers are large enough to localize the system in the vicinity of the most favoured state. Conversely, in the limit of large entropy/site that we have studied here, the barriers are too low to achieve such a result and the standard fluctuation-selection mechanism fails. We thus speculate (figure 10) that there exists a potentially ripe intermediate region between these two extreme cases where the system has a small extensive entropy and very large barriers; here the presence of glassiness in the absence of disorder remains a real possibility.

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